JOURNAL OF APPROXIMATION THEORY 49, 346–377 (1987)

Approximation to Continuous Functions by Cesàro Means of Double Fourier Series and Conjugate Series*

FERENC MÓRICZ

Bolyai Institute, University of Szeged, 6720 Szeged, Hungary

AND

XIANLIANG SHI

Department of Mathematics, University of Hangzhou, Hangzhou, People's Republic of China

Communicated by Oved Shisha

Received October 21, 1985, revised August 24, 1986

DEDICATED TO PROFESSOR PAUL R. HALMOS ON HIS 70TH BIRTHDAY

We study the rate of uniform approximation to continuous functions $f(x, y)$, 2π-periodic in each variable, in Lipschitz classes $\mathrm{Lip}(\alpha, \beta)$ and in Zygmund classes $Z(\alpha, \beta)$, $0 < \alpha, \beta \leqslant 1$, by Cesàro means $\sigma_{mn}^{\gamma\delta}(f)$ of positive orders of the rectangular partial sums of double Fourier series. The rate of uniform approximation to the conjugate functions $\tilde{f}^{(1,0)}$, $\tilde{f}^{(0,1)}$, and $\tilde{f}^{(1,1)}$ by the corresponding Cesàro means is also discussed in detail. The difference between the classes $\mathrm{Lip}(\alpha, \beta)$ and $Z(\alpha, \beta)$, similar to the one-dimensional case, appears again when $\max(\alpha, \beta) = 1$. (Compare Theorems 2 and 3 with Theorems 4 and 5.) One surprising result is the following: The uniform approximation rate by $\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)})$ to $\tilde{f}^{(1,0)}$ is worse in general than that by $\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)})$ to $\tilde{f}^{(1,1)}$ for $f \in \mathrm{Lip}(1, 1)$. In fact, the appearance of an extra factor $[\log(n+2)]^2$ in the former case is unavoidable (see Theorem 6). All approximation rates we obtain, with one exception, are shown to be exact. Two conjectures are also included.

1. INTRODUCTION: CONJUGATE SERIES AND FUNCTIONS

Let $f(x, y)$ be a complex valued function, 2π-periodic in each variable and integrable over the two-dimensional torus $-\pi < x, y \leqslant \pi$, in symbols

* This paper was written while both authors visited Indiana University, Bloomington, Indiana, U.S.A., during the summer term 1985, and while the first-named author visited the University of Wisconsin, Madison, Wisconsin, U.S.A., during the academic year 1985/86.

0021-9045/87 $3.00

$f \in L_{2\pi \times 2\pi}$. We remind the reader that the double Fourier series of f is defined by

$$S[f] = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)}, \tag{1.1}$$

where

$$c_{jk} = c_{jk}(f) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(u, v)\, e^{-i(ju+kv)}\, du\, dv \tag{1.2}$$

$(j, k = ..., -1, 0, 1, ...)$. The definitions of the three conjugate series are

$$\tilde{S}^{(1,0)}[f] = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (-i \operatorname{sign} j)\, c_{jk} e^{i(jx+ky)} \tag{1.3}$$

(*conjugate with respect to* x),

$$\tilde{S}^{(0,1)}[f] = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (-i \operatorname{sign} k)\, c_{jk} e^{i(jx+ky)} \tag{1.4}$$

(*conjugate with respect to* y), and

$$\tilde{S}^{(1,1)}[f] = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (-i \operatorname{sign} j)(-i \operatorname{sign} k)\, c_{jk} e^{i(jx+ky)} \tag{1.5}$$

(*conjugate with respect to* x *and* y). An interrelation among the series (1.1) and (1.3)–(1.5) is expressed by

$$\begin{aligned} &S[f] + i\tilde{S}^{(1,0)}[f] + i\tilde{S}^{(0,1)}[f] + i^2\tilde{S}^{(1,1)}[f] \\ &= c_{00} + 2\sum_{j=1}^{\infty} c_{j0} z^j + 2\sum_{k=1}^{\infty} c_{0k} w^k + 4\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk} z^j w^k, \end{aligned}$$

where

$$z = e^{ix} \qquad \text{and} \qquad w = e^{iy}.$$

The corresponding conjugate functions are

$$\tilde{f}^{(1,0)}(x, y) = -\frac{1}{\pi} \int_0^{\pi} [f(x+u, y) - f(x-u, y)] \frac{1}{2} \cot \frac{1}{2} u\, du, \tag{1.6}$$

$$\tilde{f}^{(0,1)}(x, y) = -\frac{1}{\pi} \int_0^{\pi} [f(x, y+v) - f(x, y-v)] \frac{1}{2} \cot \frac{1}{2} v\, dv, \tag{1.7}$$

and

$$\tilde{f}^{(1,1)}(x, y) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi [f(x+u, y+v) - f(x-u, y+v) - f(x+u, y-v) + f(x-u, y-v)] \times \tfrac{1}{2} \cot \tfrac{1}{2} u \, \tfrac{1}{2} \cot \tfrac{1}{2} v \, du \, dv. \tag{1.8}$$

In (1.6)–(1.8) the integrals are taken in the "principal value" sense at the point $u = 0$, $v = 0$, and $u = v = 0$, respectively. It follows from the corresponding one-dimensional result that if $f \in L_{2\pi \times 2\pi}$, then $\tilde{f}^{(1,0)}$ and $\tilde{f}^{(0,1)}$ exist a.e. Zygmund [10] proved that if $f \in L \log^+ L_{2\pi \times 2\pi}$, then $\tilde{f}^{(1,1)}$ also exists a.e.

Sometimes we shall write $\tilde{f}^{(1,0)})(x, y) = \tilde{f}^{(1,0)}(f, x, y)$, etc., indicating the original function whose conjugate is taken.

Let $f(x, y)$ be a continuous function, 2π-periodic in each variable, in symbols $f \in C_{2\pi \times 2\pi}$. The (partial) *moduli of continuity* of f are defined for $\delta \geqslant 0$ by

$$\omega_x(f, \delta) = \omega_{1,x}(f, \delta) = \sup_{|u| \leqslant \delta} \max_{x,y} |f(x+u, y) - f(x, y)|$$

and

$$\omega_y(f, \delta) = \omega_{1,y}(f, \delta) = \sup_{|v| \leqslant \delta} \max_{x,y} |f(x, y+v) - f(x, y)|.$$

Obviously, both $\omega_x(f, \delta)$ and $\omega_y(f, \delta)$ are nondecreasing functions.

We recall that the Lipschitz class $\mathrm{Lip}(\alpha, \beta)$, $0 < \alpha, \beta \leqslant 1$, is defined to be

$$\mathrm{Lip}(\alpha, \beta) = \{f \in C_{2\pi \times 2\pi} : \omega_x(f, \delta) = \mathcal{O}(\delta^\alpha) \text{ and } \omega_y(f, \delta) = \mathcal{O}(\delta^\beta)\}.$$

Cesari [1] proved that if $f \in \mathrm{Lip}(\alpha, \alpha)$, $0 < \alpha \leqslant 1$, then the conjugate function $\tilde{f}^{(1,1)}$ is in $\mathrm{Lip}(\alpha', \alpha')$ for each α', $0 < \alpha' < \alpha$, but need not be in $\mathrm{Lip}(\alpha, \alpha)$. An analogous statement for $\tilde{f}^{(1,0)}$ and $\tilde{f}^{(0,1)}$ was proved by Zhak [8].

The (partial) *moduli of smoothness* of a function $f \in C_{2\pi \times 2\pi}$ are defined for $\delta \geqslant 0$ by

$$\omega_{2,x}(f, \delta) = \sup_{|u| \leqslant \delta} \max_{x,y} |f(x+u, y) + f(x-u, y) - 2f(x, y)|$$

and

$$\omega_{2,y}(f, \delta) = \sup_{|v| \leqslant \delta} \max_{x,y} |f(x, y+v) + f(x, y-v) - 2f(x, y)|.$$

Clearly, both $\omega_{2,x}(f, \delta)$ and $\omega_{2,y}(f, \delta)$ are nondecreasing functions; furthermore,

$$\omega_{2,x}(f, \delta) \leqslant 2\omega_x(f, \delta) \quad \text{and} \quad \omega_{2,y}(f, \delta) \leqslant 2\omega_y(f, \delta). \tag{1.9}$$

We note that the *Zygmund class* $Z(\alpha, \beta)$ (sometimes denoted by $H^{\alpha,\beta}$), $0<0, \beta \leqslant 2$, is defined to be

$$Z(\alpha, \beta)=\{f \in C_{2\pi \times 2\pi}: \omega_{2,x}(f, \delta)=\mathcal{O}(\delta^{\alpha}) \text{ and } \omega_{2,y}(f, \delta)=\mathcal{O}(\delta^{\beta})\}.$$

It follows from (1.9) that, for $0<\alpha$, $\beta \leqslant 1$, $\operatorname{Lip}(\alpha, \beta) \subseteq Z(\alpha, \beta)$, while extending the familiar argument (see, e.g., [11, p. 44]) from the one-dimensional case to the two-dimensional case, it is easy to see that, for $0<\alpha, \beta<1$, actually $\operatorname{Lip}(\alpha, \beta)=Z(\alpha, \beta)$, and, for $0<\alpha<\beta=1$,

$$Z(\alpha, 1)=\{f \in C_{2\pi \times 2\pi}: \omega_{x}(f, \delta)=\mathcal{O}(\delta^{\alpha}) \text{ and } \omega_{2,y}(f, \delta)=\mathcal{O}(\delta)\}.$$

It is important to observe that if $f \in Z(\alpha, \beta)$, $0<\alpha, \beta \leqslant 2$, then the integrals in (1.6)–(1.8) exist in the (absolute) Lebesgue sense and thus by Fubini's theorem

$$\tilde{f}^{(1,1)}(f, x, y)=\tilde{f}^{(1,0)}(\tilde{f}^{(0,1)}, x, y)=\tilde{f}^{(0,1)}(\tilde{f}^{(1,0)}, x, y). \tag{1.10}$$

2. Approximation by Double Fourier Series

We associate with the double Fourier series (1.1) the symmetric rectangular partial sums

$$s_{mn}(f, x, y)=\sum_{j=-m}^{m} \sum_{k=-n}^{n} c_{jk}(f)\, e^{i(jx+ky)} \quad (m, n=0, 1, \ldots). \tag{2.1}$$

For $\gamma, \delta>-1$, the *Cesàro means* $\sigma_{mn}^{\gamma\delta}(f, x, y)$ *of orders* γ *and* δ, or shortly, the (C, γ, δ)*-means* of series (1.1) are defined by

$$\sigma_{mn}^{\gamma\delta}(f, x, y)=\frac{1}{A_{m}^{\gamma} A_{n}^{\delta}} \sum_{j=0}^{m} \sum_{k=0}^{n} A_{m-j}^{\gamma-1} A_{n-k}^{\delta-1} s_{jk}(f, x, y)$$
$$(m, n=0, 1, \ldots), \tag{2.2}$$

where

$$A_{m}^{\gamma}=\binom{\gamma+m}{m}=\frac{(\gamma+1)(\gamma+2) \cdots(\gamma+m)}{m!}$$

for $m=1,2, \ldots$ and $A_{0}^{\gamma}=1$. (See, e.g., [11, p. 77].)

It follows from (1.2) that

$$\sigma_{mn}^{\gamma\delta}(f, x, y)=\frac{1}{\pi^{2}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+u, y+v)\, K_{m}^{\gamma}(u)\, K_{n}^{\delta}(v)\, du\, dv, \tag{2.3}$$

where $K_m^\gamma(u)$ and $K_n^\delta(v)$ are the *Fejér kernels* in terms of u and v, respectively:

$$K_m^\gamma(u) = \frac{1}{A_m^\gamma} \sum_{j=0}^{m} A_{m-j}^{\gamma-1} D_j(u). \tag{2.4}$$

Here

$$D_j(u) = \frac{1}{2} + \sum_{\mu=1}^{j} \cos \mu u$$

is the *Dirichlet kernel*. The following representation is an easy consequence of (2.3),

$$\sigma_{mn}^{\gamma\delta}(f, x, y) - f(x, y) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \phi_{xy}(u, v)\, K_m^\gamma(u)\, K_n^\delta(v)\, du\, dv, \tag{2.5}$$

where

$$\phi_{xy}(u, v) = f(x+u, y+v) + f(x-u, y+v) + f(x+u, y-v) + f(x-u, y-v) - 4f(x, y).$$

This representation plays a crucial role in the sequel due to the estimate

$$|\phi_{xy}(u, v)| \leqslant 2\omega_{2,x}(f, u) + 2\omega_{2,y}(f, v) \qquad (u, v \geqslant 0).$$

For $f \in C_{2\pi \times 2\pi}$, let $E_{mn}(f)$ denote the best uniform approximation to f by double trigonometric polynomials $T_{mn}(x, y)$ of degree $\leqslant m$ with respect to x and of degree $\leqslant n$ with respect to y:

$$E_{mn}(f) = \inf_{T_{mn}} \| f(x, y) - T_{mn}(x, y) \|;$$

we use $\|\cdot\|$ to denote the usual maximum norm in $C_{2\pi \times 2\pi}$, i.e.,

$$\| f(x, y) - T_{mn}(x, y) \| = \max_{x, y} | f(x, y) - T_{mn}(x, y) |.$$

Analogous to the one-dimensional case, if $f \in Z(\alpha, \beta)$, $0 < \alpha, \beta \leqslant 1$, then

$$E_{mn}(f) = \mathcal{O}\left(\frac{1}{(m+1)^\alpha} + \frac{1}{(n+1)^\beta} \right).$$

This is an immediate consequence of

PROPOSITION. *If $f \in C_{2\pi \times 2\pi}$, then*

$$E_{mn}(f) = \mathcal{O}\left(\omega_{2,x}\left(f, \frac{1}{m+1} \right) + \omega_{2,y}\left(f, \frac{1}{n+1} \right) \right).$$

We are unable to find a reference to this proposition. Without entering into the details, we note that our proof closely follows that of the corresponding one-dimensional result.

The following theorem characterizes the rate of approximation to functions f in $Z(\alpha, \beta)$ by (C, γ, δ)-means of their Fourier series. The picture obtained is similar to the one-dimensional case. That is, as far as the order of magnitude is concerned, the approximation rate by $\sigma_{mn}^{\gamma\delta}(f, x, y)$ is the same as that given by $E_{mn}(f)$ is the case $\max(\alpha, \beta) < 1$, while it gets worse by a factor "log" in the case $\max(\alpha, \beta) = 1$.

THEOREM 1. *If $f \in E(\alpha, \beta)$, $0 < \alpha, \beta \leqslant 1$ and $\gamma, \delta > 0$, then*

$$\begin{aligned} \mathcal{T}_{mn}^{\gamma\delta}(f) &= \|\sigma_{mn}^{\gamma\delta}(f, x, y) - f(x, y)\| \\ &= \mathcal{O}\left(\frac{1}{(m+1)^\alpha} + \frac{1}{(n+1)^\beta}\right) && \text{if } 0 < \alpha, \beta < 1, \\ &= \mathcal{O}\left(\frac{1}{(m+1)^\alpha} + \frac{\log(n+2)}{n+1}\right) && \text{if } 0 < \alpha < \beta = 1, \\ &= \mathcal{O}\left(\frac{\log(m+2)}{m+1} + \frac{\log(n+2)}{n+1}\right) && \text{if } \alpha = \beta = 1. \end{aligned} \tag{2.6}$$

This result for $\gamma > \alpha$ and $\delta > \beta$ was proved in [4]. We emphasize that, according to (2.6), the rate of approximation does not depend on γ or δ.

The rate of approximation in each case of (2.6) is exact. This follows easily from the one-dimensional counterexamples (see, e.g., [11, p. 123]). Indeed, it suffices to take into account that if $f(x, y) = g(x) + h(y)$ with $g \in \operatorname{Lip} \alpha$ and $h \in \operatorname{Lip} \beta$, $0 < \alpha, \beta \leqslant 1$, then $f \in \operatorname{Lip}(\alpha, \beta)$ and

$$\sigma_{mn}^{\gamma\delta}(f, x, y) = \sigma_m^\gamma(g, x) + \sigma_n^\delta(h, y).$$

3. APPROXIMATION BY DOUBLE CONJUGATE SERIES

The symmetric rectangular partial sums $\tilde{s}_{mn}^{(1,0)}(f, x, y)$, $\tilde{s}^{(0,1)}(f, x, y)$, and $\tilde{s}^{(1,1)}(f, x, y)$; the Cesàro means ${}^{(1,0)}\tilde{\sigma}_{mn}^{\gamma\delta}(f, x, y)$, ${}^{(0,1)}\tilde{\sigma}_{mn}^{\gamma\delta}(f, x, y)$, and ${}^{(1,1)}\tilde{\sigma}_{mn}^{\gamma\delta}(f, x, y)$ of orders γ and δ for the double conjugate series (1.3)–(1.5) are defined analogously to (2.1) and (2.2). For instance,

$$\tilde{s}_{mn}^{(1,0)}(f, x, y) = \sum_{j=-m}^{m} \sum_{k=-n}^{n} (-i \operatorname{sign} j)\, c_{jk}(f)\, e^{i(jx+ky)}$$

and

$$^{(1,0)}\tilde{\sigma}_{mn}^{\gamma\delta}(f, x, y) = \frac{1}{A_m^\gamma A_n^\delta} \sum_{j=0}^{m} \sum_{k=0}^{n} A_{m-j}^{\gamma-1} A_{n-k}^{\delta-1} \tilde{s}_{jk}^{(1,0)}(f, x, y)$$

$$(m, n = 0, 1, \ldots).$$

The key fact is that if $f \in Z(\alpha, \beta)$ with $\alpha, \beta > 0$, then the integral in (1.6) exists in the (absolute) Lebesgue sense and by Fubini's theorem $\tilde{S}^{(1,0)}[f]$ is the double Fourier series of $\tilde{f}^{(1,0)}$, and consequently

$$\tilde{s}_{mn}^{(1,0)}(f, x, y) = s_{mn}(\tilde{f}^{(1,0)}, x, y)$$

and

$$^{(1,0)}\tilde{\sigma}_{mn}^{\gamma\delta}(f, x, y) = \sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)}, x, y).$$

From now on, we shall use the notations indicated in the right-hand sides of these equalities. Similar observations pertain to $\tilde{S}^{(0,1)}[f]$ and $\tilde{S}^{(1,1)}[f]$.

The following two theorems characterize the rate of approximation to the conjugate functions $\tilde{f}^{(1,0)}$ and $\tilde{f}^{(1,1)}$ by the (C, γ, δ)-means of the corresponding conjugate series for functions $f \in Z(\alpha, \beta)$. We will not deal with $\tilde{f}^{(0,1)}$ separately. All theorems on $\tilde{f}^{(1,0)}$ can be reformulated for $\tilde{f}^{(0,1)}$ by taking their symmetric counterparts.

Different from the one-dimensional case, the approximation rate both by $\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)})$ to $\tilde{f}^{(1,0)}$ and by $\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)})$ to $\tilde{f}^{(1,1)}$ is always worse at least by a factor "log" than the one that occurs in $E_{mn}(f)$. More precisely, the approximation rate by $\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)})$ to $\tilde{f}^{(1,0)}$ contains an extra factor $\log(m+2)$ only if $\alpha = 1$, but it does contain an extra factor $\log(n+2)$ if $\beta < 1$ and even an extra factor $[\log(n+2)]^2$ if $\beta = 1$. On the other hand, the approximation rate by $\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)})$ to $\tilde{f}^{(1,1)}$ symmetrically contains an extra "log" factor in both m and n if $\max(\alpha, \beta) < 1$, and an extra "$\log^2$" factor in both m and n if $\alpha = \beta = 1$.

THEOREM 2. *If $f \in Z(\alpha, \beta)$, $0 < \alpha, \beta \leqslant 1$, and $\gamma, \delta > 0$, then*

$$\begin{aligned}
\mathscr{T}_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)}) &= \|\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)}, x, y) - \tilde{f}^{(1,0)}(x, y)\| \\
&= \mathcal{O}\left(\frac{1}{(m+1)^\alpha} + \frac{\log(n+2)}{(n+1)^\beta}\right) && \textit{if } 0 < \alpha, \beta < 1, \\
&= \mathcal{O}\left(\frac{\log(m+2)}{m+1} + \frac{\log(n+2)}{(n+1)^\beta}\right) && \textit{if } 0 < \beta < \alpha = 1, \\
&= \mathcal{O}\left(\frac{1}{(m+1)^\alpha} + \frac{[\log(n+2)]^2}{n+1}\right) && \textit{if } 0 < \alpha < \beta = 1, \\
&= \mathcal{O}\left(\frac{\log(m+2)}{m+1} + \frac{[\log(n+2)]^2}{n+1}\right) && \textit{if } \alpha = \beta = 1. \qquad (3.1)
\end{aligned}$$

THEOREM 3. *If $f \in Z(\alpha, \beta)$, $0 < \alpha$, $\beta \leqslant 1$, and γ, $\delta > 0$, then*

$$\begin{aligned}
\mathcal{T}_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)}) &= \|\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)}, x, y) - \tilde{f}^{(1,1)}(x, y)\| \\
&= \mathcal{O}\left(\frac{\log(m+2)}{(m+1)^{\alpha}} + \frac{\log(n+2)}{(n+1)^{\beta}}\right) \qquad \textit{if } 0 < \alpha, \beta < 1, \\
&= \mathcal{O}\left(\frac{\log(m+2)}{(m+1)^{\alpha}} + \frac{[\log(n+2)]^2}{n+1}\right) \qquad \textit{if } 0 < \alpha < \beta = 1, \\
&= \mathcal{O}\left(\frac{(\log(m+2)]^2}{m+1} + \frac{[\log(n+2)]^2}{n+1}\right) \qquad \textit{if } \alpha = \beta = 1.
\end{aligned} \tag{3.2}$$

Theorems 2 and 3 were proved in [5] for the cases $\alpha > \gamma$ and $\beta > \delta$. Again the rate of approximation does not depend on γ or δ.

Both Theorems 2 and 3 can be improved for functions $f \in \text{Lip}(\alpha, \beta)$, $0 < \alpha$, $\beta \leqslant 1$, in the cases where $\alpha = 1$ and $\max(\alpha, \beta) = 1$, respectively. The approximation rate by $\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)})$ to $\tilde{f}^{(1,0)}$ drops the factor $\log(m+2)$ for $\alpha = 1$, but the factor $\log(n+2)$ does remain for $\beta < 1$, and, surprisingly, the factor $[\log(n+2)]^2$ is unavoidable for $\beta = 1$. On the other hand, the approximation rate by $\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)})$ to $\tilde{f}^{(1,1)}$ contains only "log" factors in both m and n for all α and β, independently of whether $\alpha, \beta < 1$ or $= 1$.

THEOREM 4. *If $f \in \text{Lip}(\alpha, \beta)$, $0 < \alpha$, $\beta \leqslant 1$ and γ, $\delta > 0$, then*

$$\begin{aligned}
\mathcal{T}_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)}) &= \mathcal{O}\left(\frac{1}{(m+1)^{\alpha}} + \frac{\log(n+2)}{(n+1)^{\beta}}\right) \qquad \textit{if } 0 < \beta < 1, \\
&= \mathcal{O}\left(\frac{1}{(m+1)^{\alpha}} + \frac{[\log(n+2)]^2}{n+1}\right) \qquad \textit{if } \beta = 1.
\end{aligned} \tag{3.3}$$

THEOREM 5. *If $f \in \text{Lip}(\alpha, \beta)$, $0 < \alpha$, $\beta \leqslant 1$ and γ, $\delta > 0$, then*

$$\mathcal{T}_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)}) = \mathcal{O}\left(\frac{\log(m+2)}{(m+1)^{\alpha}} + \frac{\log(n+2)}{(n+1)^{\beta}}\right). \tag{3.4}$$

The rate of approximation in Theorems 4 and 5 is exact. This is shown by the next two theorems.

THEOREM 6. *There exist functions $f = f_{\alpha\beta} \in \text{Lip}(\alpha, \beta)$, $0 < \alpha$, $\beta \leqslant 1$, such that for any γ, $\delta > 0$ neither of the estimates*

$$\begin{aligned}\mathcal{T}^{\gamma\delta}_{mn}(\mathcal{J}^{(1,0)}) &= o\left(\frac{1}{(m+1)^{\alpha}}\right) + \mathcal{O}(\lambda(n)) && \textit{if } 0<\alpha\leqslant 1,\\ &= \mathcal{O}(\lambda(m)) + o\left(\frac{\log(n+2)}{(n+1)^{\beta}}\right) && \textit{if } 0<\beta<1,\\ &= \mathcal{O}(\lambda(m)) + o\left(\frac{[(\log(n+2)]^2}{n+1}\right) && \textit{if } \beta=1, \end{aligned} \tag{3.5}$$

can hold.

Here and in the sequel, by $\{\lambda(m): m=0, 1,...\}$ we denote an arbitrary nonincreasing sequence of positive numbers tending to 0.

THEOREM 7. *There exist functions $f=f_{\alpha}\in \text{Lip}(\alpha, 1)$, $0<\alpha\leqslant 1$, such that for any γ, $\delta>0$ the estimate*

$$\mathcal{T}^{\gamma\delta}_{mn}(\mathcal{J}^{(1,1)}) = o\left(\frac{\log(m+2)}{(m+1)^{\alpha}}\right) + \mathcal{O}(\lambda(n)) \tag{3.6}$$

cannot hold.

From Theorems 6 and 7 it follows immediately that estimates (3.1)(i) and (3.1)(iii) in Theorem 2 as well as estimate (3.2)(i) and the first part of (3.2)(ii) are exact.

Finally, we show that estimates (3.1)(ii) and (3.1)(iv) are also exact; that is, Theorem 2 is best possible.

THEOREM 8. *There exists a function $f\in Z(1, 1)$ such that for any γ, $\delta>0$ the estimate*

$$\mathcal{T}^{\gamma\delta}_{mn}(\mathcal{J}^{(1,0)}) = o\left(\frac{\log(m+2)}{m+1}\right) + \mathcal{O}(\lambda(n)) \tag{3.7}$$

cannot hold.

The only estimate whose exactness we are unable to prove is (3.2)(iii).

Conjecture 1. There exists a function $f\in Z(1, 1)$ such that for any γ, $\delta>0$ the estimate

$$\mathcal{T}^{\gamma\delta}_{mn}(\mathcal{J}^{(1,1)}) = o\left(\frac{[\log(m+2)]^2}{m+1}\right) + \mathcal{O}(\lambda(n))$$

cannot hold.

4. Auxiliary Results

Let $g(x)$ be a 2π-periodic continuous function of a single variable, in symbols $g \in C_{2\pi}$. We shall use the following notations:

$\tilde{g}(x)$: the conjugate function to $g(x)$,

$\sigma_m^\gamma(g, x)$: the mth (C, γ)-mean of the partial sums of the Fourier series of g,

$E_m(g)$: the best approximation to g by trigonometric polynomials of degree $\leqslant m$,

$\omega(g, \delta) = \omega_1(g, \delta)$: the modulus of continuity of g,

$\omega_2(g, \delta)$: the modulus of smoothness of g.

In the sequel, we need several well-known results from the approximation theory of periodic continuous functions in one variable. For the sake of convenience, we list them as follows:

(a) *Generalized Privalov's theorem* (see, e.g., [7, pp. 162–163, 391]). If

$$\int_0^\delta \frac{\omega(g, t)}{t}\, dt = \mathcal{O}(\omega(g, \delta)) \qquad \text{as} \quad \delta \to +0,$$

then

$$\omega(\tilde{g}, \delta) = \mathcal{O}(\omega(g, \delta)).$$

In particular, if

$$\omega(g, \delta) = \mathcal{O}(\delta^\alpha), \qquad 0 < \alpha \leqslant 1,$$

then

$$\begin{aligned} \omega(\tilde{g}, \delta) &= \mathcal{O}(\delta^\alpha) && \text{if} \quad 0 < \alpha < 1, \\ &= \mathcal{O}(\delta \log \tfrac{1}{\delta}) && \text{if} \quad \alpha = 1. \end{aligned}$$

All these statements remain valid if "o" is substituted for "$\mathcal{O}$" both in the conditions and in the conclusions.

(b) *Zygmund's theorem* [9]. If

$$\omega_2(g, \delta) = \mathcal{O}\left(\delta^\alpha \log^\varepsilon \frac{1}{\delta}\right) \qquad 0 < \alpha < 2, \quad \varepsilon \geqslant 0,$$

then

$$\omega_2(\tilde{g}, \delta) = \mathcal{O}\left(\delta^\alpha \log^\varepsilon \frac{1}{\delta}\right).$$

(c) Relation between the moduli of continuity and of smoothness (see, e.g., [7, p. 107]). If

$$\omega_2(g, \delta) = \mathcal{O}(\delta^\alpha), \qquad 0 < \alpha < 1,$$

then

$$\begin{aligned} \omega(g, \delta) &= \mathcal{O}(\delta^\alpha) && \text{if} \quad 0 < \alpha < 1, \\ &= \mathcal{O}\left(\delta \log \frac{1}{\delta}\right) && \text{if} \quad \alpha = 1. \end{aligned}$$

(d) *Alexits' theorem* (see, e.g., [11, p. 123]). $\tilde{g} \in \operatorname{Lip} 1$ if and only if

$$\|\sigma_m^1(g, x) - g(x)\| = \mathcal{O}\left(\frac{1}{m+1}\right),$$

where $\|\cdot\|$ is the usual "max" norm in $C_{2\pi}$.

(e) For the *Fejér kernel* $K_m^\gamma(u)$ (see (2.4)) we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} K_m^\gamma(u)\, du = 1 \qquad \text{if} \quad \gamma > -1$$

and

$$\int_{-\pi}^{\pi} |K_m^\gamma(u)|\, du = \mathcal{O}(1) \qquad \text{if} \quad \gamma > 0;$$

(see, e.g., [11, p. 94]).

(f) *Jackson's theorem* (see, e.g., [7, p. 260]).

$$E_m(g) = \mathcal{O}\left(\omega_2\left(g, \frac{1}{m+1}\right)\right).$$

(g) *Stechkin's inequalities* [6; 7, p. 331].

$$\omega\left(g, \frac{1}{m+1}\right) = \mathcal{O}\left(\frac{1}{m+1} \sum_{j=0}^{m} E_j(g)\right)$$

and

$$\omega_2\left(g, \frac{1}{m+1}\right) = \mathcal{O}\left(\frac{1}{(m+1)^2} \sum_{j=0}^{m} (j+1)\, E_j(g)\right).$$

(h) Continuity of $\tilde{g}$ (see, e.g., [7, p. 319]). If

$$\sum_{j=1}^{\infty} \frac{E_j(g)}{j} < \infty,$$

then $\tilde{g} \in C_{2\pi}$ and

$$E_m(\tilde{g}) = \mathcal{O}\left(E_m(g) + \sum_{j=m+1}^{\infty} \frac{E_j(g)}{j}\right).$$

Now we present a number of lemmas. The first of them is known in the literature, while the others are new.

LEMMA 1. *If $g \in C_{2\pi}$ and $\gamma > 0$, then*

$$\sigma_m^{\gamma}(g, x) - g(x) = \frac{\gamma}{\pi(m+1)} \int_{1/(m+1)}^{\infty} \frac{g(x+u) + g(x-u) - 2g(x)}{u^2}\, du + \mathcal{O}\left(\omega_2\left(g, \frac{1}{m+1}\right)\right), \tag{4.1}$$

with "$\mathcal{O}$" independent of m, x, and g.

This estimate was proved by Efimov [2] for $\gamma = 1$ and by Guo [3] for $\gamma > 0$.

The following lemma expresses an important consequence of Lemma 1 often used in the sequel. In particular, it is used to prove Theorems 4 and 5.

LEMMA 2. *If $g \in \operatorname{Lip} 1$, then*

$$\int_{1/(m+1)}^{\infty} \frac{\tilde{g}(x+u) + \tilde{g}(x-u) - 2\tilde{g}(x)}{u^2}\, du = \mathcal{O}(1). \tag{4.2}$$

Remark. In (4.2) we can equally well take the integral $\int_{1/(m+1)}^{\pi}$ instead of $\int_{1/(m+1)}^{\infty}$ because of the absolute convergence of $\int_{\pi}^{\infty} (\tilde{g}(x+u) + \tilde{g}(x-u) - 2\tilde{g}(x))\, u^{-2}\, du$.

Proof of Lemma 2. Applying (4.1) for $\tilde{g}$ and $\gamma = 1$ yields

$$\sigma_m^{1}(\tilde{g}, x) - \tilde{g}(x) = \frac{1}{\pi(m+1)} \int_{1/(m+1)}^{\infty} \frac{\tilde{g}(x+u) + \tilde{g}(x-u) - 2\tilde{g}(x)}{u^2}\, du + \mathcal{O}\left(\omega_2\left(\tilde{g}, \frac{1}{m+1}\right)\right). \tag{4.3}$$

Clearly $g \in \operatorname{Lip} 1$ implies $\omega_2(g, \delta) = \mathcal{O}(\delta)$ and, by Zygmund's theorem, $\omega_2(\tilde{g}, \delta) = \mathcal{O}(\delta)$. Consequently,

$$\omega_2\left(\tilde{g}, \frac{1}{m+1}\right) = \mathcal{O}\left(\frac{1}{m+1}\right). \tag{4.4}$$

On the other hand, by Alexits' theorem

$$\|\sigma_m^1(\tilde{g}, x) - \tilde{g}(x)\| = \mathcal{O}\left(\frac{1}{m+1}\right). \tag{4.5}$$

Combining (4.3)–(4.5) provides (4.2).

The next lemma is a natural extension of Lemma 1 to functions in two variables and will be the main tool in the proofs of Theorems 1–5.

LEMMA 3. *If* $f \in C_{2\pi \times 2\pi}$ *and* γ, $\delta > 0$, *then*

$$\begin{aligned}
&\sigma_{mn}^{\gamma\delta}(f, x, y) - f(x, y)\\
&\quad = \frac{\gamma\delta}{\pi^2(m+1)(n+1)} \int_{1/(m+1)}^{\infty} \int_{1/(n+1)}^{\infty} \frac{du}{u^2} \frac{dv}{v^2}\\
&\qquad \times \{[f(x+u, y+v) + f(x-u, y+v) - 2f(x, y+v)]\\
&\qquad + [f(x+u, y-v) + f(x-u, y-v) - 2f(x, y-v)]\\
&\qquad - 2[f(x+u, y) + f(x-u, y) - 2f(x, y)]\}\\
&\qquad + \frac{\gamma}{\pi(m+1)} \int_{1/(m+1)}^{\infty} \frac{f(x+u, y) + f(x-u, y) - 2f(x, y)}{u^2}\, du\\
&\qquad + \frac{\delta}{\pi(n+1)} \int_{1/(n+1)}^{\infty} \frac{f(x, y+v) + f(x, y-v) - 2f(x, y)}{v^2}\, dv\\
&\qquad + \mathcal{O}\left(\omega_{2,x}\left(f, \frac{1}{m+1}\right) + \omega_{2,y}\left(f, \frac{1}{n+1}\right)\right),
\end{aligned} \tag{4.6}$$

with "$\mathcal{O}$" independent of m, n, x, y, *and* f.

Before sketching the proof of this lemma, we introduce two notations we use frequently later on. Given a function $f(x, y)$ we can consider f as a function of x for each fixed y, as well as a function of y for each fixed x denoted, respectively, by

$$g_y(x) = f(x, y) \qquad \text{and} \qquad h_x(y) = f(x, y).$$

Obviously, if $f \in C_{2\pi \times 2\pi}$, then $g_y \in C_{2\pi}$ for every y and

$$\tilde{g}_y(x) = \tilde{f}^{(1,0)}(x, y);$$

furthermore, $h_x \in C_{2\pi}$ for every x and

$$\tilde{h}_x(y) = \tilde{f}^{(0,1)}(x, y).$$

Proof of Lemma 3. We use representation (2.5) and the corresponding one-dimensional representation

$$\sigma_m^\gamma(g_y, x) - g_y(x) = \frac{1}{\pi}\int_0^\pi [g_y(x+u) + g_y(x-u) - 2g_y(x)]\, K_m^\gamma(u)\, du$$
$$= \frac{1}{\pi}\int_0^\pi [f(x+u, y) + f(x-u, y) - 2f(x, y)]\, K_m^\gamma(u)\, du.$$

On the basis of these representations we can deduce the recurrent relation

$$\sigma_{mn}^{\gamma\delta}(f, x, y) - f(x, y) = [\sigma_n^\delta(\sigma_m^\gamma(g_y, x) - g_y(x), y) - (\sigma_m^\gamma(g_y, x) - g_y(x))]$$
$$+ [\sigma_m^\gamma(g_y, x) - g_y(x)] + [\sigma_n^\delta(h_x, y) - h_x(y)]. \quad (4.7)$$

Now, (4.6) is a consequence of (4.7) and of the repeated application of (4.1). We do not go into further details.

The following lemma will be useful in the proofs of Theorems 6 and 7.

LEMMA 4. *If $f \in C_{2\pi \times 2\pi}$ and γ, $\delta > 0$, then for every fixed m,*

$$\lim_{n \to \infty} \|\sigma_{mn}^{\gamma\delta}(f, x, y) - \sigma_m^\gamma(g_y, x)\| = 0 \quad (4.8)$$

and for every fixed n,

$$\lim_{m \to \infty} \|\sigma_{mn}^{\gamma\delta}(f, x, y) - \sigma_n^\delta(h_x, y)\| = 0. \quad (4.9)$$

Proof. We will prove only (4.8). Using representation (2.3) and the corresponding one-dimensional representation

$$\sigma_m^\gamma(g_y, x) = \frac{1}{\pi}\int_{-\pi}^\pi g_y(x+u)\, K_m^\gamma(u)\, du$$
$$= \frac{1}{\pi^2}\int_{-\pi}^\pi \int_{-\pi}^\pi f(x+u, y)\, K_m^\gamma(u)\, K_n^\delta(v)\, du\, dv,$$

we get

$$\sigma_{mn}^{\gamma\delta}(f, x, y) - \sigma_m^\gamma(g_y, x)$$
$$= \frac{1}{\pi}\int_{-\pi}^\pi K_m^\gamma(u) \left\{\frac{1}{\pi}\int_{-\pi}^\pi [f(x+u, y+v) - f(x+u, y)]\, K_n^\delta(v)\, dv\right\} du$$
$$= \frac{1}{\pi}\int_{-\pi}^\pi K_m^\gamma(u)[\sigma_n^\delta(h_{x+u}, y) - h_{x+u}(y)]\, du,$$

whence, by (e),

$$\begin{aligned}\|\sigma_{mn}^{\gamma\delta}(f, x, y) - \sigma_m^{\gamma}(g_y, x)\| &\leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |K_m^{\gamma}(u)| \cdot \|\sigma_n^{\delta}(h_{x+u}, y) - h_{x+u}(y)\|\, du \\ &= \mathcal{O}(1)\|\sigma_n^{\delta}(h_x, y) - h_x(y)\| = o(1) \qquad \text{as} \quad n \to \infty,\end{aligned}$$

which is (4.8) to be proved. Here the estimate $o(1)$ is the well-known Fejér–Riesz theorem (see, e.g., [11, pp. 94–95]) while, taking into account that $f \in C_{2\pi \times 2\pi}$ is uniformly continuous in x and y, it follows that $o(1)$ does not depend on x and y but merely on δ. Similarly, $\mathcal{O}(1)$ depends only on γ.

If $f \in \mathrm{Lip}(\alpha, \beta)$, $0 < \alpha$, $\beta \leqslant 1$, then the conjugate functions $\tilde{f}^{(1,0)}$, $\tilde{f}^{(0,1)}$, and $\tilde{f}^{(1,1)}$ need not belong to the same $\mathrm{Lip}(\alpha, \beta)$ (see [1, 8] for the case where $\alpha = \beta$), but they are in a function class close to it as Lemma 5 shows.

LEMMA 5. *If $f \in \mathrm{Lip}(\alpha, \beta)$, $0 < \alpha$, $\beta \leqslant 1$, then*

$$\begin{aligned}\omega_x(\tilde{f}^{(1,0)}, \delta) &= \mathcal{O}(\delta^{\alpha}) && \text{if} \quad 0 < \alpha < 1, \\ &= \mathcal{O}\left(\delta \log \frac{1}{\delta}\right) && \text{if} \quad \alpha = 1;\end{aligned} \tag{4.10}$$

$$\omega_y(\tilde{f}^{(1,0)}, \delta) = \mathcal{O}\left(\delta^{\beta} \log \frac{1}{\delta}\right); \tag{4.11}$$

$$\begin{aligned}\omega_x(\tilde{f}^{(1,1)}, \delta) &= \mathcal{O}\left(\delta^{\alpha} \log \frac{1}{\delta}\right) && \text{if} \quad 0 < \alpha < 1, \\ &= \mathcal{O}\left(\delta \log^2 \frac{1}{\delta}\right) && \text{if} \quad \alpha = 1.\end{aligned} \tag{4.12}$$

The corresponding estimates for $\omega_x(\tilde{f}^{(0,1)}, \delta)$, $\omega_y(\tilde{f}^{(0,1)}, \delta)$, and $\omega_y(\tilde{f}^{(1,1)}, \delta)$ are the symmetric counterparts of (4.11), (4.10), and (4.12), respectively.

It follows from the corresponding one-dimensional counterexamples that estimate (4.10) is exact, and it follows from Lemma 7 below that estimates (4.11) and (4.12)(i) are also exact. The only estimate whose exactness we are unable to prove is (4.12)(ii).

Conjecture 2. There exists a function $f \in \mathrm{Lip}(1,1)$ such that the estimate

$$\omega_x(\tilde{f}^{(1,1)}, \delta) = o\left(\delta \log^2 \frac{1}{\delta}\right) \qquad (\delta \to +0)$$

cannot hold.

Proof of Lemma 5. Estimate (4.10) can be obtained via Privalov's theorem applied to $g_y(x)=f(x, y)$.

In order to prove (4.11), let $v>0$ and choose ε such that $\varepsilon\alpha=\beta$. Splitting both integrals into two parts gives

$$\begin{aligned}
&\tilde{f}^{(1,0)}(x, y+v)-\tilde{f}^{(1,0)}(x, y)\\
&\quad=-\frac{1}{\pi}\int_0^{v^\varepsilon}[f(x+u, y+v)-f(x-u, y+v)]\frac{1}{2}\cot\frac{1}{2}u\,du\\
&\quad\quad+\frac{1}{\pi}\int_0^{v^\varepsilon}[f(x+u, y)-f(x-u, y)]\frac{1}{2}\cot\frac{1}{2}u\,du\\
&\quad\quad-\frac{1}{\pi}\int_{v^\varepsilon}^{\pi}[f(x+u, y+v)-f(x+u, y)]\frac{1}{2}\cot\frac{1}{2}u\,du\\
&\quad\quad+\frac{1}{\pi}\int_{v^\varepsilon}^{\pi}[f(x-u, y+v)-f(x-u, y)]\frac{1}{2}\cot\frac{1}{2}u\,du, \qquad (4.13)
\end{aligned}$$

whence

$$\begin{aligned}
&|\tilde{f}^{(1,0)}(x, y+v)-\tilde{f}^{(1,0)}(x, y)|\\
&\quad=\frac{2}{\pi}\int_0^{v^\varepsilon}\frac{\mathcal{O}(u^\alpha)}{u}\,du+\frac{2}{\pi}\int_{v^\varepsilon}^{\pi}\frac{\mathcal{O}(v^\beta)}{u}\,du\\
&\quad=\mathcal{O}(v^{\varepsilon\alpha})+\mathcal{O}\left(v^\beta\log\frac{1}{v}\right)=\mathcal{O}\left(v^\beta\log\frac{1}{v}\right).
\end{aligned}$$

This results in (4.11).

Now we turn to the proof of (4.12). The crucial relation is (1.10). Let $u>0$ and choose η such that $\eta\beta=\alpha$. Similar to (4.13),

$$\begin{aligned}
&\tilde{f}^{(1,1)}(x+u, y)-\tilde{f}^{(1,1)}(x, y)\\
&\quad=-\frac{1}{\pi}\int_0^{\pi}[\tilde{f}^{(1,0)}(x+u, y+v)-\tilde{f}^{(1,0)}(x+u, y-v)]\frac{1}{2}\cot\frac{1}{2}v\,dv\\
&\quad\quad+\frac{1}{\pi}\int_0^{\pi}[\tilde{f}^{(1,0)}(x, y+v)-\tilde{f}^{(1,0)}(x, y-v)]\frac{1}{2}\cot\frac{1}{2}v\,dv\\
&\quad=-\frac{1}{\pi}\int_0^{u^\eta}[\tilde{f}^{(1,0)}(x+u, y+v)-\tilde{f}^{(1,0)}(x+u, y-v)]\frac{1}{2}\cot\frac{1}{2}v\,dv\\
&\quad\quad+\frac{1}{\pi}\int_0^{u^\eta}[\tilde{f}^{(1,0)}(x, y+v)-\tilde{f}^{(1,0)}(x, y-v)]\frac{1}{2}\cot\frac{1}{2}v\,dv\\
&\quad\quad-\frac{1}{\pi}\int_{u^\eta}^{\pi}[\tilde{f}^{(1,0)}(x+u, y+v)-\tilde{f}^{(1,0)}(x, y+v)]\frac{1}{2}\cot\frac{1}{2}v\,dv\\
&\quad\quad+\frac{1}{\pi}\int_{u^\eta}^{\pi}[\tilde{f}^{(1,0)}(x+u,y-v)-\tilde{f}^{(1,0)}(x, y-v)]\frac{1}{2}\cot\frac{1}{2}v\,dv. \quad (4.14)
\end{aligned}$$

Hence for $0<\alpha<1$, by (4.10) and (4.11),

$$\begin{aligned}
&|\tilde{f}^{(1,1)}(x+u, y)-\tilde{f}^{(1,1)}(x, y)| \\
&\quad =\frac{2}{\pi}\int_0^{u^\eta} \frac{\mathcal{O}(v^\beta \log \frac{1}{v})}{v}\, dv+\frac{2}{\pi}\int_{u^\eta}^{\pi} \frac{\mathcal{O}(u^\alpha)}{v}\, dv \\
&\quad =\mathcal{O}\left(u^\alpha \log \frac{1}{u}\right)+\mathcal{O}\left(u^\alpha \log \frac{1}{u}\right)=\mathcal{O}\left(u^\alpha \log \frac{1}{u}\right),
\end{aligned}$$

while for $\alpha=1$,

$$\begin{aligned}
&|\tilde{f}^{(1,1)}(x+u, y)-\tilde{f}^{(1,1)}(x, y)| \\
&\quad =\frac{2}{\pi}\int_0^{u^\eta} \frac{\mathcal{O}(v^\beta \log \frac{1}{v})}{v}\, dv+\frac{2}{\pi}\int_{u^\eta}^{\pi} \frac{\mathcal{O}(u \log \frac{1}{u})}{v}\, dv \\
&\quad =\mathcal{O}\left(u^{\eta\beta} \log \frac{1}{u}\right)+\mathcal{O}\left(u \log^2 \frac{1}{u}\right)=\mathcal{O}\left(u \log^2 \frac{1}{u}\right),
\end{aligned}$$

proving (4.12).

The moduli of smoothness of the conjugate functions $\tilde{f}^{(1,0)}$, $\tilde{f}^{(0,1)}$, and $\tilde{f}^{(1,1)}$ exhibit a nicer behavior than the corresponding moduli of continuity, in accordance with the one-dimensional experience.

LEMMA 6. *If $f \in Z(\alpha, \beta)$ and $0<\alpha, \beta<2$, then*

$$\omega_{2,x}(\tilde{f}^{(1,0)}, \delta)=\mathcal{O}(\delta^\alpha), \tag{4.15}$$

$$\omega_{2,y}(\tilde{f}^{(1,0)}, \delta)=\mathcal{O}\left(\delta^\beta \log \frac{1}{\delta}\right), \tag{4.16}$$

$$\omega_{2,x}(\tilde{f}^{(1,1)}, \delta)=\mathcal{O}\left(\delta^\alpha \log \frac{1}{\delta}\right). \tag{4.17}$$

We note that the corresponding estimates for $\omega_{2,x}(\tilde{f}^{(0,1)}, \delta)$, $\omega_{2,y}(\tilde{f}^{(0,1)}, \delta)$ and $\omega_{2,y}(\tilde{f}^{(1,1)}, \delta)$ are the symmetric counterparts of (4.16), (4.15), and (4.17), respectively.

Proof of Lemma 6. First, we prove that if

$$\omega_{2,x}(f, \delta)=\mathcal{O}(\delta^\alpha), \qquad 0<\alpha<2,$$

then (4.15) holds. This is a consequence of Privalov's theorem if $0<\alpha<1$ and of Zygmund's theorem if $\alpha=1$. In the general case $0<\alpha<2$, we apply (g), (h), and (f) to $\tilde{g}_y(x)=\tilde{f}^{(1,0)}(x, y)$ in order to obtain

$$\omega_{2,x}\left(\tilde{g}_y, \frac{1}{m+1}\right) = \mathcal{O}\left(\frac{1}{(m+1)^2}\sum_{j=0}^{m}(j+1)\,E_j(\tilde{g}_y)\right)$$
$$= \mathcal{O}\left\{\frac{1}{(m+1)^2}\sum_{j=0}^{m}(j+1)\,E_j(g_y) + \frac{1}{(m+1)^2}\sum_{j=0}^{m}(j+1)\sum_{k=j+1}^{\infty}\frac{E_k(g_y)}{k}\right\}$$
$$= \mathcal{O}\left(\frac{1}{(m+1)^{\alpha}}\right).$$

This proves (4.15).

Second, we prove (4.16). Let $v > 0$ and choose ε such that $\varepsilon \min(\alpha, 1) = \beta$. Similar to (4.13),

$$\tilde{f}^{(1,0)}(x, y+v) + \tilde{f}^{(1,0)}(x, y-v) - 2\tilde{f}^{(1,0)}(x, y)$$
$$= -\frac{1}{\pi}\int_0^{v^\varepsilon}[f(x+u, y+v) - f(x-u, y+v)]\,\frac{1}{2}\cot\frac{1}{2}u\,du$$
$$-\frac{1}{\pi}\int_0^{v^\varepsilon}[f(x+u, y-v) - f(x-u, y-v)]\,\frac{1}{2}\cot\frac{1}{2}u\,du$$
$$+\frac{2}{\pi}\int_0^{v^\varepsilon}[f(x+u, y) - f(x-u, y)]\,\frac{1}{2}\cot\frac{1}{2}u\,du$$
$$-\frac{1}{\pi}\int_{v^\varepsilon}^{\pi}[f(x+u, y+v) + f(x+u, y-v) - 2f(x+u, y)]\,\frac{1}{2}\cot\frac{1}{2}u\,du$$
$$+\frac{1}{\pi}\int_{v^\varepsilon}^{\pi}[f(x-u, y+v) + f(x-u, y-v) - 2f(x-u, y)]\,\frac{1}{2}\cot\frac{1}{2}u\,du, \tag{4.18}$$

whence

$$|\tilde{f}^{(1,0)}(x, y+v) + \tilde{f}^{(1,0)}(x, y-v) - 2\tilde{f}^{(1,0)}(x, y)|$$
$$= \frac{4}{\pi}\int_0^{v^\varepsilon}\frac{\mathcal{O}(\omega_{1,x}(f, u))}{u}\,du + \frac{2}{\pi}\int_{v^\varepsilon}^{\pi}\frac{\mathcal{O}(\omega_{2,y}(f, v))}{u}\,du. \tag{4.19}$$

By (c),

$$\int_0^{v^\varepsilon}\frac{\mathcal{O}(\omega_{1,x}(f, u))}{u}\,du$$
$$= \mathcal{O}(v^{\varepsilon\alpha}) = \mathcal{O}(v^{\beta}) \qquad \text{if} \quad 0 < \alpha < 1,$$
$$= \mathcal{O}\left(v^{\varepsilon}\log\frac{1}{v}\right) = \mathcal{O}\left(v^{\beta}\log\frac{1}{v}\right) \qquad \text{if} \quad 1 \leqslant \alpha < 2. \tag{4.20}$$

Clearly,

$$\int_{v^{\alpha}}^{\pi} \frac{\mathcal{O}(\omega_{2,y}(f, v))}{u}\, du = \mathcal{O}\left(v^{\beta} \log \frac{1}{v}\right). \tag{4.21}$$

Combining (4.19)–(4.21) yields (4.16).

Third, we prove (4.17). Let $u > 0$ and fix $\bar{\beta}$ such that $0 < \bar{\beta} < \min(\beta, 1)$. Then choose η such that $v\bar{\beta} = \alpha$. Using representation (1.10) and arguing similarly to (4.14) and (4.18) results in

$$\begin{aligned} &|\tilde{f}^{(1,1)}(x+u, y) + \tilde{f}^{(1,1)}(x-u, y) - 2\tilde{f}^{(1,1)}(x, y)| \\ &\quad = \frac{4}{\pi} \int_0^{u^\eta} \frac{\mathcal{O}(\omega_{1,y}(\tilde{f}^{(1,0)}, v))}{v}\, dv + \frac{2}{\pi} \int_{u^\eta}^{\pi} \frac{\mathcal{O}(\omega_{2,x}(\tilde{f}^{(1,0)}, u))}{v}\, dv. \end{aligned}$$

By (4.16) and the choice of $\bar{\beta}$,

$$\omega_{2,y}(\tilde{f}^{(1,0)}, v) = \mathcal{O}(v^{\bar{\beta}}),$$

whence by (c),

$$\omega_{1,y}(\tilde{f}^{(1,0)}, v) = \mathcal{O}(v^{\bar{\beta}}).$$

Now, we can complete the proof of (4.17) in the same manner as above in the case of the proof of (4.16).

We use Lemmas 5 and 6 in the proofs of Theorems 2–5. The next lemma is of basic importance in proving Theorems 6 and 7. To prove Theorem 8 we provide a direct counterexample.

LEMMA 7. *Let $\omega_1(\delta)$ and $\omega_2(\delta)$ be two moduli of continuity; ω_1 is strictly increasing at a certain right-hand neighborhood of $\delta = 0$. Define*

$$x_k = \omega_1^{-1}(\omega_2(2^{-k-2})) \qquad (k \geqslant k_0), \tag{4.22}$$

where ω_1^{-1} denotes the inverse function of ω_1 at a right-hand neighborhood of $\delta = 0$. Assume that $x_{k_0} < 1$ and

$$x_k \leqslant C_1 x_{k+1} \qquad (k \geqslant k_0), \tag{4.23}$$

with a constant C_1 and an integer k_0.

Then there exists a function $f \in C_{2\pi \times 2\pi}$ such that

$$\omega_{1,x}(f, \delta) = \mathcal{O}(\omega_1(\delta)), \tag{4.24}$$

$$\omega_{1,y}(f, \delta) = \mathcal{O}(\omega_2(\delta)), \tag{4.25}$$

and

$$\omega_{2,y}(\tilde{f}^{(1,0)}, 2^{-k-2}) \geqslant C_2 \omega_2(2^{-k-2}) \log \frac{1}{x_k} \qquad (k \geqslant k_0), \tag{4.26}$$

with a constant $C_2 > 0$.

It is clear that if

$$\omega_1(\delta)=\delta^{\alpha} \quad \text{and} \quad \omega_2(\delta)=\delta^{\beta}, \qquad 0<\alpha, \beta \leqslant 1,$$

then x_k is defined for every $k=0, 1,...$ and condition (4.23) is satisfied (but it may happen that $x_k \geqslant 1$ for finitely many values of k). In this particular case, by (4.24)–(4.26) there exists a function $f \in \mathrm{Lip}(\alpha, \beta)$ such that

$$\omega_{2,y}(\tilde{f}^{(1,0)}, 2^{-k-2}) \geqslant C_3(k+1)\, 2^{-\beta k} \qquad (k=0, 1,...)$$

with another constant $C_3>0$. We state this corollary in the form of

LEMMA 8. *There exist functions* $f=f_\beta \in \mathrm{Lip}(1, \beta)$, $0<\beta \leqslant 1$, *such that the estimate*

$$\omega_{2,y}(\tilde{f}^{(1,0)}, \delta)=o\left(\delta^{\beta} \log \frac{1}{\delta}\right) \tag{4.27}$$

cannot hold.

The symmetric counterpart of Lemma 8 says that there exist (possibly other) functions $f=f_\alpha \in \mathrm{Lip}(\alpha, 1)$, $0<\alpha \leqslant 1$, such that the estimate

$$\omega_{2,x}(\tilde{f}^{(0,1)}, \delta)=o\left(\delta^{\alpha} \log \frac{1}{\delta}\right) \tag{4.28}$$

cannot hold.

According to (4.27) and (4.28), estimates (4.11) and (4.12)(i) as well as (4.16) and (4.17) are the best possible.

Proof of Lemma 7. Without loss of generality, we may assume that $\omega_1(\delta)$ is strictly increasing for every $\delta \geqslant 0$, that $x_0<1$, and that x_k is a strictly decreasing sequence for $k=0, 1,...$.

Define $f \in C_{2\pi \times 2\pi}$ as

$$\begin{aligned}
f(x, y) &= 0 && \text{if } -\pi \leqslant x \leqslant \pi,\ -\pi \leqslant y \leqslant 0, \text{ or } \tfrac{5}{4} \leqslant y \leqslant \pi \text{ or} \\
& && \quad 2^{-k-1}+2^{-k-3} \leqslant y \leqslant 2^{-k}-2^{-k-2}; \\
&= 0 && \text{if } -\pi \leqslant x \leqslant -x_k,\ y=2^{-k}; \\
&= \omega_2(2^{-k-2})\frac{x+x_k}{x_k} && \text{if } -x_k \leqslant x \leqslant 0,\ y=2^{-k}; \\
&= \omega_2(2^{-k-2}) && \text{if } 0 \leqslant x \leqslant \pi-x_k,\ y=2^{-k}; \\
&= \omega_2(2^{-k-2})\frac{\pi-x}{x_k} && \text{if } \pi-x_k \leqslant x \leqslant \pi,\ y=2^{-k};
\end{aligned}$$

where $k=0, 1, \ldots$ in each case; and $f(x, y)$ is defined by means of a linear interpolation for each x, $-\pi \leqslant x \leqslant \pi$, and for each of the intervals

$$2^{-k}-2^{-k-2} \leqslant y \leqslant 2^{-k} \quad \text{and} \quad 2^{-k} \leqslant y \leqslant 2^{-k}+2^{-k-2} \quad (k=0, 1, \ldots).$$

First, we will check (4.24). To this effect, let $-\pi \leqslant x$, $y \leqslant \pi$ and $u>0$ be given and we estimate $|f(x+u, y)-f(x, y)|$. We may assume that $0<y<\frac{5}{4}$ and $u<x_0$; in particular,

$$2^{-l}-2^{-l-2} \leqslant y < 2^{-l}+2^{-l-2} \tag{4.29}$$

and

$$x_{k+1} \leqslant u < x_k$$

with some k, $l=0, 1, \ldots$.

If $k \leqslant l$, then the "worst" case occurs where $y=2^{-k}$:

$$\begin{aligned} &|f(x+u, y)-f(x, y)| \\ &\quad \leqslant \omega_2(2^{-k-2}) = \omega_1(\omega_1^{-1}(\omega_2(2^{-k-2}))) = \omega_1(x_k) \\ &\quad \leqslant \omega_1(C_1 x_{k+1}) \leqslant \omega_1(C_1 u) = \mathcal{O}(\omega_1(u)). \end{aligned}$$

If $k>l$, then the "worst" case occurs where $y=2^{-l}$:

$$\begin{aligned} &|f(x+u, y)-f(x, y)| \\ &\quad \leqslant \omega_2(2^{-l-2}) \frac{u}{x_l} = \frac{u}{x_l}\, \omega_1(\omega_1^{-1}(\omega_2(2^{-l-2}))) = \frac{u}{x_l}\, \omega_1(x_l) \\ &\quad \leqslant \frac{u}{x_l} \left[\frac{x_l}{u}+1\right] \omega_1(u) = \mathcal{O}(\omega_1(u)). \end{aligned}$$

Second, we verify (4.25). To this end, we estimate $|f(x, y+v)-f(x, y)|$, where $-\pi \leqslant x$, $y \leqslant \pi$ and $v>0$ are given. We may assume that $0<y<\frac{5}{4}$ and $v<\frac{1}{4}$, in particular,

$$2^{-k-3} \leqslant v < 2^{-k-2} \tag{4.30}$$

with some $k=0, 1, \ldots$.

If $y \leqslant 2^{-k}+2^{-k-2}$, then clearly

$$|f(x, y+v)-f(x, y)| \leqslant \omega_2(2^{-k-2}) \leqslant 2\omega_2(v).$$

If (4.29) is the case with $l<k$, then by (4.30),

$$\begin{aligned} &|f(x, y+v)-f(x, y)| \\ &\quad \leqslant \omega_2(2^{-l-2})\, 2^{l+2} v \leqslant \omega_2(2^{-l-2})\, 2^{l-k} = 2^{l-k} \omega_2(2^{k-l} 2^{-k-2}) \\ &\quad \leqslant \omega_2(2^{-k-2}) \leqslant 2\omega_2(v). \end{aligned}$$

Third, we will show (4.26). Let $x = 0$,

$$w_k = 2^{-k} - 2^{-k-2}, \quad y_k = 2^{-k}, \quad \text{and} \quad z_k = 2^{-k} + 2^{-k-2} \qquad (k = 0, 1, \ldots).$$

By definition,

$$f(u, w_k) = f(u, z_k) = 0 \qquad \text{for every } u,$$

consequently,

$$\tilde{f}^{(1,0)}(0, w_k) = \tilde{f}^{(1,0)}(0, z_k) = 0.$$

On the other hand,

$$\begin{aligned} \tilde{f}^{(1,0)}(0, y_k) &= -\frac{1}{\pi}\int_0^{\pi} [f(u, y_k) - f(-u, y_k)] \frac{1}{2}\cot\frac{1}{2}u\,du \\ &= -\frac{1}{\pi}\int_0^{x_k} \omega_2(2^{-k-2})\frac{u}{x_k}\frac{1}{2}\cot\frac{1}{2}u\,du \\ &\quad -\frac{1}{\pi}\int_{x_k}^{\pi - x_k} \omega_2(2^{-k-2})\frac{1}{2}\cot\frac{1}{2}u\,du \\ &\quad -\frac{1}{\pi}\int_{\pi - x_k}^{\pi} \omega_2(2^{-k-2})\frac{\pi - x}{x_k}\frac{1}{2}\cot\frac{1}{2}u\,du, \end{aligned}$$

whence

$$|\tilde{f}^{(1,0)}(0, y_k)| \geqslant C_4\omega_2(2^{-k-2})\left(\log\frac{1}{x_k} - O(1)\right), \tag{4.31}$$

with a constant $C_4 > 0$. Now, (4.26) follows from (4.31) and the trivial estimate

$$\begin{aligned} \omega_{2,y}(\tilde{f}^{(1,0)}, 2^{-k-2}) &\geqslant |\tilde{f}^{(1,0)}(0, w_k) + \tilde{f}^{(1,0)}(0, z_k) - 2\tilde{f}^{(1,0)}(0, y_k)| \\ &= 2|\tilde{f}^{(1,0)}(0, y_k)|. \end{aligned}$$

5. The Saturation Problem

It is easy to see that if for a function $f \in C_{2\pi \times 2\pi}$ we have

$$\|\sigma_{mn}^{\gamma\delta}(f, x, y) - f(x, y)\| = o\left(\frac{1}{m+1} + \frac{1}{n+1}\right)$$

with certain fixed γ, $\delta > 0$, then necessarily $f = \text{constant}$ (cf. the corresponding one-dimensional result in [11, p. 122]). In other words, the operator $\sigma^{\gamma\delta}_{mn}(f)$ is saturable with the *saturation order* $\{1/(m+1)+1/(n+1)\}$. The collection of functions

$$\mathscr{F}=\left\{f\in C_{2\pi\times 2\pi}:\|\sigma^{\gamma\delta}_{mn}(f,x,y)-f(x,y)\|=\mathcal{O}\left(\frac{1}{m+1}+\frac{1}{n+1}\right)\right\}$$

is called the *saturation class* of $\sigma^{\gamma\delta}_{mn}(f)$. Now, we can easily show that $f\in C_{2\pi\times 2\pi}$ is a saturation function if and only if

$$\omega_x(\tilde{f}^{(1,0)},\delta)=\mathcal{O}(\delta)\quad\text{and}\quad\omega_y(\tilde{f}^{(0,1)},\delta)=\mathcal{O}(\delta). \tag{5.1}$$

In fact, if $f\in\mathscr{F}$, then letting $n\to\infty$, by Lemma 4 we get for every y,

$$\|\sigma^{\gamma}_{m}(g_y,x)-g_y(x)\|=\mathcal{O}\left(\frac{1}{m+1}\right)$$

and "$\mathcal{O}$" is independent of y. From Alexits' theorem, the first inequality in (5.1) follows. The second one can be obtained analogously.

Conversely, assume (5.1) holds. Then thanks to relation (4.7), Alexits' theorem, and the continuity of the operator $\sigma^{\delta}_{n}(h)$, we can conclude eventually that $f\in\mathscr{F}$.

In a similar manner, we can verify that the operators $\sigma^{\gamma\delta}_{mn}(\tilde{f}^{(1,0)})$ and $\sigma^{\gamma\delta}_{mn}(\tilde{f}^{(1,1)})$ are also saturable with the saturation order $\{1/(m+1)+1/(n+1)\}$. Denote by $\tilde{\mathscr{F}}^{(1,0)}$ and $\tilde{\mathscr{F}}^{(1,1)}$ the corresponding saturation classes. Then $f\in\tilde{\mathscr{F}}^{(1,0)}$ if and only if

$$\omega_x(f,\delta)=\mathcal{O}(\delta)\quad\text{and}\quad\omega_y(\tilde{f}^{(1,1)},\delta)=\mathcal{O}(\delta), \tag{5.2}$$

while $f\in\tilde{\mathscr{F}}^{(1,1)}$ if and only if

$$\omega_x(\tilde{f}^{(0,1)},\delta)=\mathcal{O}(\delta)\quad\text{and}\quad\omega_y(\tilde{f}^{(1,0)},\delta)=\mathcal{O}(\delta). \tag{5.3}$$

6. Proofs of Theorems 1–5

In each case, the key formula is (4.6) in Lemma 3.

Proof of Theorem 1. Since $\omega_{2,x}(f,\delta)=\mathcal{O}(\delta^\alpha)$, we have

$$\begin{aligned}\int_{1/(m+1)}^{\pi}&\frac{f(x+u,y)+f(x-u,y)-2f(x,y)}{u^2}\,du\\&=\mathcal{O}(1)\int_{1/(m+1)}^{\pi}\frac{\mathcal{O}(u^\alpha)}{u^2}\,du\\&=\mathcal{O}((m+1)^{1-\alpha})\qquad\text{if}\quad 0<\alpha<1,\\&=\mathcal{O}(\log(m+2))\qquad\text{if}\quad\alpha=1.\end{aligned}\tag{6.1}$$

Similarly, $\omega_{2,y}(f, \delta) = \mathcal{O}(\delta^\beta)$ and

$$\int_{1/(n+1)}^{\infty} \frac{f(x, y+v) + f(x, y-v) - 2f(x, y)}{v^2} \, dv$$
$$= \mathcal{O}((n+1)^{1-\beta}) \quad \text{if} \quad 0 < \beta < 1,$$
$$= \mathcal{O}(\log(n+2)) \quad \text{if} \quad \beta = 1; \tag{6.2}$$

and

$$\int_{1/(m+1)}^{\infty} \int_{1/(n+1)}^{\infty} \frac{du}{u^2} \frac{dv}{v^2} \left[f(x+u, y+v) + f(x-u, y+v) - 2f(x, y+v)\right]$$
$$= \mathcal{O}(1) \int_{1/(m+1)}^{\pi} \frac{\mathcal{O}(u^\alpha)}{u^2} \, du \int_{1/(n+1)}^{\infty} \frac{dv}{v^2}$$
$$= \mathcal{O}((n+1)(m+1)^{1-\alpha}) \quad \text{if} \quad 0 < \alpha < 1,$$
$$= \mathcal{O}((n+1) \log(m+2)) \quad \text{if} \quad \alpha = 1; \tag{6.3}$$

and two more similar estimates.

Combining (4.6) and (6.1)–(6.3) yields (2.6).

Proof of Theorem 2. This time we make use of Lemma 6 together with Lemma 3 (the latter applied to $\tilde{f}^{(1,0)}$). Analogously to (6.1), by (4.15) we have

$$\int_{1/(m+1)}^{\infty} \frac{\tilde{f}^{(1,0)}(x+u, y) + \tilde{f}^{(1,0)}(x-u, y) - 2\tilde{f}^{(1,0)}(x, y)}{u^2} \, du$$
$$= \mathcal{O}((m+1)^{1-\alpha}) \quad \text{if} \quad 0 < \alpha < 1,$$
$$= \mathcal{O}(\log(m+2)) \quad \text{if} \quad \alpha = 1. \tag{6.4}$$

By (4.16),

$$\int_{1/(n+1)}^{\infty} \frac{\tilde{f}^{(1,0)}(x, y+v) + \tilde{f}^{(1,0)}(x, y-v) - 2\tilde{f}^{(1,0)}(x, y)}{v^2} \, dv$$
$$= \mathcal{O}(1) \int_{1/(n+1)}^{\pi} \frac{\mathcal{O}(v^\beta \log 1/v)}{v^2} \, dv$$
$$= \mathcal{O}((n+1)^{1-\beta} \log(n+2)) \quad \text{if} \quad 0 < \beta < 1,$$
$$= ([\log(n+2)]^2) \quad \text{if} \quad \beta = 1; \tag{6.5}$$

and three more similar estimates.

Collecting (4.6), (4.15), (4.16), (6.4), and (6.5) provides (3.1).

Proof of Theorem 3. It is essentially a repetition of that of Theorem 2, with the exception that this time we apply Lemma 3 to $\tilde{f}^{(1,1)}$ and use (4.17) and its symmetric counterpart instead of (4.15) and (4.16).

Proof of Theorem 4. For $0<\alpha<1$ and $0<\beta\leqslant 1$, estimates (3.1) and (3.3) coincide. Thus, we have to prove (3.3) in the case where $\alpha=1$ and $0<\beta\leqslant 1$. To this goal, we will apply Lemma 3 to $\tilde{f}^{(1,0)}$ while taking Lemma 2 into account.

To go into details, by assumption $g_y(x)\in \operatorname{Lip} 1$ for every y, where $g_y(x)=f(x,y)$. Therefore, by (4.2)

$$\frac{\gamma}{\pi(m+1)}\int_{1/(m+1)}^{\pi}\frac{\tilde{f}^{(1,0)}(x+u,y)+\tilde{f}^{(1,0)}(x-u,y)-2\tilde{f}^{(1,0)}(x,y)}{u^2}\,du$$
$$=\frac{\gamma}{\pi(m+1)}\int_{1/(m+1)}^{\pi}\frac{\tilde{g}_y(x+u)+\tilde{g}_y(x-u)-2\tilde{g}(x)}{u^2}\,du+\mathcal{O}\left(\frac{1}{m+1}\right), \qquad (6.6)$$

where "$\mathcal{O}$" is independent of x and y. Estimate (6.6) clearly remains true if y is replaced in turn by $y+v$ and $y-v$.

Due to (4.16), a simple computation gives

$$\frac{\delta}{\pi(n+1)}\int_{1/(n+1)}^{\pi}\frac{\tilde{f}^{(1,0)}(x,y+v)+\tilde{f}^{(1,0)}(x,y-v)-2\tilde{f}^{(1,0)}(x,y)}{v^2}\,dv$$
$$=\frac{\delta}{\pi(n+1)}\int_{1/(n+1)}^{\pi}\frac{\mathcal{O}(v^{\beta}\log 1/v)}{v^2}\,dv$$
$$=\mathcal{O}\left(\frac{\log(n+2)}{(n+1)^{\beta}}\right) \qquad \text{if} \quad 0<\beta<1,$$
$$=\mathcal{O}\left(\frac{[\log(n+2)]^2}{n+1}\right) \qquad \text{if} \quad \beta=1, \qquad (6.7)$$

where "$\mathcal{O}$" is independent of x and y.

Finally, we apply Lemma 3 to $\tilde{f}^{(1,0)}$. Then by (4.6), (6.6) (with y, $y+v$, and $y-v$), (6.7), (4.15), and (4.16) we get

$$|\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)},x,y)-\tilde{f}^{(1,0)}(x,y)|$$
$$=\frac{4\delta}{\pi(n+1)}\int_{1/(n+1)}^{\pi}\frac{dv}{v^2}\,\mathcal{O}\left(\frac{1}{m+1}\right)$$
$$+\mathcal{O}\left(\frac{1}{m+1}\right)+\begin{cases}\mathcal{O}\left(\dfrac{\log(n+2)}{(n+1)^{\beta}}\right) & \text{if} \quad 0<\beta<1\\[2ex] \mathcal{O}\left(\dfrac{[\log(n+2)]^2}{n+1}\right) & \text{if} \quad \beta=1\end{cases}$$
$$+\mathcal{O}\left(\frac{1}{m+1}+\frac{\log(n+2)}{(n+1)^{\beta}}\right)$$

$$= \mathcal{O}\left(\frac{1}{m+1}\right) + \begin{Bmatrix} \mathcal{O}\left(\dfrac{\log(n+2)}{(n+1)^{\beta}}\right) & \text{if} \quad 0<\beta<1 \\ \mathcal{O}\left(\dfrac{[\log(n+2)]^2}{n+1}\right) & \text{if} \quad \beta=1 \end{Bmatrix},$$

where the "$\mathcal{O}$" are independent of x and y. This proves (3.3) in the case $\alpha=1$ and $0<\beta\leqslant 1$.

Proof of Theorem 5. For $0<\alpha$, $\beta<1$ estimates (3.2) and (3.4) coincide. Also we have to prove (3.4) in the case where $\max(\alpha, \beta)=1$. For example, let $\alpha=1$ and $0<\beta\leqslant 1$.

We introduce an auxiliary function G_t defined by

$$G_t(x, y)=\int_t^{\infty} \frac{\tilde{f}^{(1,0)}(x+u, y)+\tilde{f}^{(1,0)}(x-u, y)-2\tilde{f}^{(1,0)}(x, y)}{u^2}\, du. \tag{6.8}$$

By assumption $f\in \operatorname{Lip}(1, \beta)$, and thus $g_y(x)=f(x, y)\in \operatorname{Lip} 1$ for every y. Applying Lemma 2 to g_y results in the crucial estimate

$$G_t(x, y)=\mathcal{O}(1) \qquad \text{as} \quad t\to +0, \tag{6.9}$$

where "$\mathcal{O}$" is uniform in x and y.

We will consider the conjugate function to G_t with respect to y:

$$\tilde{G}_t^{(0,1)}(x, y)= -\frac{1}{\pi}\int_0^{\pi} [G_t(x, y+v)-G_t(x, y-v)]\, \frac{1}{2}\cot\frac{1}{2}v\, dv.$$

On the one hand, keeping (6.8) in mind, $\tilde{G}_t^{(0,1)}$ can be estimated in the following way. Let $\varepsilon=1/\beta$ and decompose the integral $\int_0^{\pi}=\int_0^{t^{\varepsilon}}+\int_{t^{\varepsilon}}^{\pi}$ with respect to v:

$$\begin{aligned}
&\tilde{G}_t^{(0,1)}(x, y)\\
&\quad= -\frac{1}{\pi}\int_0^{t^{\varepsilon}} \frac{1}{2}\cot\frac{1}{2}v\, dv \int_t^{\infty} \frac{du}{u^2}\\
&\qquad\times \{[\tilde{f}^{(1,0)}(x+u, y+v)-\tilde{f}^{(1,0)}(x+u, y-v)]\\
&\qquad+[\tilde{f}^{(1,0)}(x-u, y+v)-\tilde{f}^{(1,0)}(x-u, y-v)]\\
&\qquad-2[\tilde{f}^{(1,0)}(x, y+v)-\tilde{f}^{(1,0)}(x, y-v)]\}\\
&\qquad-\frac{1}{\pi}\int_{t^{\varepsilon}}^{\pi} \frac{1}{2}\cot\frac{1}{2}v\, dv \int_t^{\infty} \frac{du}{u^2}\\
&\qquad\times \{[\tilde{f}^{(1,0)}(x+u, y+v)+\tilde{f}^{(1,0)}(x-u, y+v)-2\tilde{f}^{(1,0)}(x, y+v)]\\
&\qquad-[\tilde{f}^{(1,0)}(x+u, y-v)+\tilde{f}^{(1,0)}(x-u, y-v)-2\tilde{f}^{(1,0)}(x, y-v)]\}\\
&\quad=J_1+J_2, \qquad \text{say.}
\end{aligned} \tag{6.10}$$

By (4.11),

$$|J_1| = \frac{4}{\pi}\int_0^{t^\varepsilon} \frac{dv}{v}\, \mathcal{O}\left(v^\beta \log\frac{1}{v}\right)\int_t^\infty \frac{du}{u^2}$$
$$= \frac{1}{t}\int_0^{t^\varepsilon} \mathcal{O}\left(v^{\beta-1}\log\frac{1}{v}\right) dv$$
$$= \frac{1}{t}\,\mathcal{O}\left(t^{\varepsilon\beta}\log\frac{1}{t}\right) = \mathcal{O}\left(\log\frac{1}{t}\right) \qquad \text{as} \quad t\to +0. \tag{6.11}$$

To estimate J_2, the decisive fact is that the inner integral is $\mathcal{O}(1)$ due to Lemma 2. So

$$|J_2| \leqslant \frac{2}{\pi}\int_{t^\varepsilon}^{\pi} \frac{dv}{v}\,\mathcal{O}(1) = \mathcal{O}\left(\log\frac{1}{t}\right) \qquad \text{as} \quad t\to +0. \tag{6.12}$$

To sum up, (6.11) and (6.12) yield

$$\tilde{G}_t^{(0,1)}(x, y) = \mathcal{O}\left(\log\frac{1}{t}\right) \qquad \text{as} \quad t\to +0, \tag{6.13}$$

where the "$\mathcal{O}$" are independent of x and y.

On the other hand, by Fubini's theorem (cf. (6.6))

$$\tilde{G}_t^{(0,1)}(x, y)$$
$$= \int_t^\infty \frac{du}{u^2}\left(-\frac{1}{\pi}\int_0^\pi [\tilde{f}^{(1,0)}(x+u, y+v) - \tilde{f}^{(1,0)}(x+u, y-v)]\,\frac{1}{2}\cot\frac{1}{2}v\, dv\right.$$
$$-\frac{1}{\pi}\int_0^\pi [\tilde{f}^{(1,0)}(x-u, y+v) - \tilde{f}^{(1,0)}(x-u, y-v)]\,\frac{1}{2}\cot\frac{1}{2}v\, dv$$
$$\left.+\frac{2}{\pi}\int_0^\pi [\tilde{f}^{(1,0)}(x, y+v) - \tilde{f}^{(1,0)}(x, y-v)]\,\frac{1}{2}\cot\frac{1}{2}v\, dv\right)$$
$$= \int_t^\infty \frac{\tilde{f}^{(1,1)}(x+u, y) + \tilde{f}^{(1,1)}(x-u, y) - 2\tilde{f}^{(1,1)}(x, y)}{u^2}\, du. \tag{6.14}$$

Combining (6.13) and (6.14) results in

$$\frac{\gamma}{\pi(m+1)}\int_{1/(m+1)}^\infty \frac{\tilde{f}^{(1,1)}(x+u, y) + \tilde{f}^{(1,1)}(x-u, y) - 2\tilde{f}^{(1,1)}(x, y)}{u^2}\, du$$
$$= \frac{\gamma}{\pi(m+1)}\,\tilde{G}_{1/(m+1)}^{(0,1)}(x, y) = \mathcal{O}\left(\frac{\log(m+2)}{m+1}\right), \tag{6.15}$$

where "$\mathcal{O}$" is independent of x and y.

Similarly, in case $\beta = 1$ we have

$$\frac{\delta}{\pi(n+1)} \int_{1/(n+1)}^{\infty} \frac{\tilde{f}^{(1,1)}(x, y+v) + \tilde{f}^{(1,1)}(x, y-v) - 2\tilde{f}^{(1,1)}(x, y)}{v^2}\, dv$$

$$= \mathcal{O}\left(\frac{\log(n+2)}{n+1}\right), \tag{6.16}$$

while in case $0 < \beta < 1$ we have (making use of the symmetric counterpart of (4.17))

$$\frac{\delta}{\pi(n+1)} \int_{1/(n+1)}^{\infty} \frac{\tilde{f}^{(1,1)}(x, y+v) + \tilde{f}^{(1,1)}(x, y-v) - 2\tilde{f}^{(1,1)}(x, y)}{v^2}\, dv$$

$$\leqslant \frac{\mathcal{O}(1)}{n+1} \int_{1/(n+1)}^{\infty} \mathcal{O}(v^{\beta-2} \log v)\, dv = \mathcal{O}\left(\frac{\log(n+2)}{(n+1)^{\beta}}\right), \tag{6.17}$$

where "$\mathcal{O}$" is independent of x and y.

Now we apply Lemma 3 to $\tilde{f}^{(1,1)}$. On account of (4.6), (6.15)–(6.17) (clearly, (6.15) holds true if y is replaced in turn by $y+v$ and $y-v$), (4.17), and its symmetric counterpart, we can conclude that

$$\begin{aligned}
&|\sigma_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)}, x, y) - \tilde{f}^{(1,1)}(x, y)| \\
&\quad = \frac{4\delta}{\pi(n+1)} \int_{1/(n+1)}^{\infty} \frac{dv}{v^2}\, \mathcal{O}\left(\frac{\log(m+2)}{m+1}\right) + \mathcal{O}\left(\frac{\log(m+2)}{m+1}\right) \\
&\quad\quad + \mathcal{O}\left(\frac{\log(n+2)}{(n+1)^{\beta}}\right) + \mathcal{O}\left(\frac{\log(m+2)}{m+1} + \frac{\log(n+2)}{(n+1)^{\beta}}\right) \\
&\quad = \mathcal{O}\left(\frac{\log(m+2)}{m+1} + \frac{\log(n+2)}{(n+1)^{\beta}}\right),
\end{aligned}$$

where the "$\mathcal{O}$" are uniform in x and y. This proves (3.4) in the case where $\alpha = 1$ and $0 < \beta \leqslant 1$.

The proof in the case $0 < \alpha \leqslant 1$ and $\beta = 1$ is similar and therefore we omit it.

7. Proofs of Theorems 6–8

As before, we set $g_y(x) = f(x, y)$. Then

$$\tilde{g}_y(x) = \tilde{f}^{(1,0)}(x, y)$$

and

$$\tilde{\tilde{g}}_y(x) = \widetilde{\tilde{f}^{(1,0)}(x, y)}^{(1,0)} = -f(x, y) + f(0, y). \tag{7.1}$$

Proof of Theorem 6. If (3.5)(i) holds, then letting $n \to \infty$ via Lemma 4, we get for every fixed y

$$\sigma_m^{\gamma}(\tilde{g}_y, x) - \tilde{g}_y(x) = o\left(\frac{1}{(m+1)^{\alpha}}\right). \tag{7.2}$$

If $\alpha = 1$, this implies that $\tilde{g}_y(x) =$ constant depending on y, since $1/(m+1)$ is the saturation order. This is a contradiction.

Now we deal with the case $0 < \alpha < 1$. From (7.2) it follows that

$$E_m(\tilde{g}_y) = o\left(\frac{1}{(m+1)^{\alpha}}\right),$$

which implies in turn by (g)

$$\omega_x(\tilde{g}_y, \delta) = o(\delta^{\alpha}),$$

and by (a),

$$\omega_x(\tilde{\tilde{g}}_y, \delta) = o(\delta^{\alpha}).$$

According to (7.1), the latter inequality is equivalent to

$$\omega_x(f, \delta) = o(\delta^{\alpha})$$

which is impossible in general.

Second, assume that (3.5)(ii) holds. Letting $m \to \infty$ via Lemma 4, we get for every fixed x

$$\sigma_n^{\delta}(h_x, y) - h_x(y) = o\left(\frac{\log(n+2)}{(n+1)^{\beta}}\right),$$

where this time we set $h_x(y) = \tilde{f}^{(1,0)}(x, y)$ with $f \in \text{Lip}(\alpha, \beta)$, $0 < \beta < 1$, from Lemma 8. This yields

$$E_n(h_x) = o\left(\frac{\log(n+2)}{(n+1)^{\beta}}\right),$$

whence by (g)

$$\omega_y(h_x, \delta) = o\left(\delta^{\beta} \log \frac{1}{\delta}\right)$$

which is equivalent to

$$\omega_y(\tilde{f}^{(1,0)}, \delta) = o\left(\delta^{\beta} \log \frac{1}{\delta}\right).$$

The latter estimate cannot hold owing to Lemma 8.

Third, assume that (3.5)(iii) holds. Letting again $m \to \infty$, via Lemma 4 we get for every fixed x,

$$\sigma_n^\delta(h_x, y) - h_x(y) = o\left(\frac{[\log(n+2)]^2}{n+1}\right) \tag{7.3}$$

with another $h_x(y) = \tilde{f}^{(1,0)}(x, y)$, where $f \in \mathrm{Lip}(1, 1)$ will be defined later on. We apply Lemma 1 to $h_x \in C_{2\pi}$ and $\delta > 0$. By (4.1),

$$\begin{aligned}
&\sigma_n^\delta(h_x, y) - h_x(y) \\
&\quad = \frac{\delta}{\pi(n+1)} \int_{1/(n+1)}^{\infty} \frac{h_x(y+v) + h_x(y-v) - 2h_x(y)}{v^2}\, dv \\
&\qquad + \mathcal{O}\left(\omega_{2,y}\left(h_x, \frac{1}{n+1}\right)\right),
\end{aligned}$$

whence, on account of (7.3), we get that

$$\begin{aligned}
&\frac{\delta}{\pi(n+1)} \int_{1/(n+1)}^{\infty} \frac{\tilde{f}^{(1,0)}(x, y+v) + \tilde{f}^{(1,0)}(x, y-v) - 2\tilde{f}^{(1,0)}(x, y)}{v^2}\, dv \\
&\qquad + \mathcal{O}\left(\omega_{2,y}\left(\tilde{f}^{(1,0)}, \frac{1}{n+1}\right)\right) = o\left(\frac{[\log(n+2)]^2}{n+1}\right).
\end{aligned} \tag{7.4}$$

By (4.11), we have

$$\omega_{2,y}\left(\tilde{f}^{(1,0)}, \frac{1}{n+1}\right) = \mathcal{O}\left(\frac{\log(n+2)}{n+1}\right).$$

Combining this inequality with (7.4) yields

$$\begin{aligned}
J_t(x, y) &= -\int_t^{\pi} \frac{\tilde{f}^{(1,0)}(x, y+v) + \tilde{f}^{(1,0)}(x, y-v) - 2\tilde{f}^{(1,0)}(x, y)}{v^2}\, dv \\
&= o\left(\log^2 \frac{1}{t}\right) \qquad \text{as} \quad t \to +0.
\end{aligned} \tag{7.5}$$

On the other hand, consider the function $f \in C_{2\pi \times 2\pi}$ defined by

$$\begin{aligned}
f(x, y) &= \min(x, y, \pi - x, \pi - y) && \text{if } 0 \leqslant x,\ y \leqslant \pi, \\
&= 0 && \text{if } -\pi \leqslant x \leqslant 0,\ -\pi \leqslant y \leqslant \pi, \text{ or} \\
& && \quad 0 \leqslant x \leqslant \pi,\ -\pi \leqslant y \leqslant 0.
\end{aligned}$$

It is obvious that $f \in \mathrm{Lip}(1, 1)$. Setting $x = 0$, by definition,

$$\tilde{f}^{(1,0)}(0, y) = -\frac{1}{\pi} \int_0^{\pi} f(u, y) \frac{1}{2} \cot \frac{1}{2} u\, du.$$

Setting also $y=0$,

$$J_t(0,0) = -\int_t^\pi \frac{\tilde{f}^{(1,0)}(0,v)}{v^2}\,dv$$

$$= \frac{1}{\pi}\int_t^\pi \int_0^t \frac{f(u,v)}{v^2}\frac{1}{2}\cot\frac{1}{2}u\,dv\,du,$$

whence

$$J_t(0,0) \geqslant C_5 \int_t^{\pi/2}\int_t^{\pi/2} \frac{dv}{v}\frac{du}{u} = C_5\left[\log\frac{\pi}{2t}\right]^2 \tag{7.6}$$

with a constant $C_5>0$. Clearly, (7.5) and (7.6) contradict each other.

Proof of Theorem 7. Assume that (3.6) holds with the $f \in \mathrm{Lip}(\alpha, 1)$ from the symmetric counterpart of Lemma 8, i.e., for which (4.28) is not satisfied. Set $\varphi_y(x) = \tilde{f}^{(1,1)}(x,y)$ for every fixed y. Then by letting $n\to\infty$ in (3.6), due to Lemma 4, we get that

$$\sigma_m^\gamma(\varphi_y, x) - \varphi_y(x) = o\left(\frac{\log(m+2)}{(m+1)^\alpha}\right).$$

Hence obviously

$$E_m(\varphi_y) = o\left(\frac{\log(m+2)}{(m+1)^\alpha}\right)$$

and, by the second Stechkin inequality,

$$\omega_{2,x}(\tilde{f}^{(1,1)}, \delta) = \omega_2(\varphi_y, \delta) = o\left(\delta^\alpha \log\frac{1}{\delta}\right).$$

Since

$$\tilde{\varphi}_y(x) = \widetilde{\tilde{f}^{(1,1)}(x,y)}^{(1,0)} = -\tilde{f}^{(0,1)}(x,y) + \tilde{f}^{(0,1)}(0,y)$$

(cf. (7.1)), we can conclude, using (a), that

$$\omega_{2,x}(\tilde{f}^{(0,1)}, \delta) = \omega_2(\tilde{\varphi}_y, \delta) = o\left(\delta^\alpha \log\frac{1}{\delta}\right).$$

But the inequality just obtained cannot hold according to the symmetric counterpart of Lemma 8.

Proof of Theorem 8. We set $f(x,y) = g(x)$, where $g \in C_{2\pi}$ is defined by the condition that

$$\tilde{g}(x) = |x| - \frac{\pi}{2} \qquad \text{if} \quad -\pi \leqslant x \leqslant \pi,$$

i.e., $g = -\tilde{\tilde{g}}$ (if we require in addition that $g(0) = 0$). Clearly $\tilde{g} \in Z(1)$, so by (b) (with $\varepsilon = 0$) we also have $g \in Z(1)$, and thus $f \in Z(1, 1)$.

Now if (3.7) holds, then letting $n \to \infty$, via Lemma 4 we get that

$$\sigma_m^{\gamma}(\tilde{g}, x) - \tilde{g}(x) = o\left(\frac{\log(m+2)}{m+1}\right). \tag{7.7}$$

Putting $x = 0$, and using Lemma 2,

$$\begin{aligned}
&\sigma_m^{\gamma}(\tilde{g}, 0) - \tilde{g}(0) \\
&= \frac{\gamma}{\pi(m+1)} \int_{1/(m+1)}^{\pi} \frac{\tilde{g}(u) + \tilde{g}(-u) - 2\tilde{g}(0)}{u^2}\, du + \mathcal{O}\left(\omega_2\left(\tilde{g}, \frac{1}{m+1}\right)\right) \\
&= \frac{\gamma}{\pi(m+1)} \int_{1/(m+1)}^{\pi} \frac{2u}{u^2}\, du + \mathcal{O}\left(\frac{1}{m+1}\right) \\
&= \frac{2\gamma}{\pi(m+1)} \log \pi(m+1) + \mathcal{O}\left(\frac{1}{m+1}\right).
\end{aligned}$$

This contradicts (7.7) and the proof of Theorem 8 is complete.

References

1. L. Cesari, Sulle serie di Fourier delle funzioni lipschitziane di più variabili, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2)*. **7** (1938), 279–295.
2. A. V. Efimov, On approximation of certain classes of continuous functions by Fourier sums and Fejér sums, *Izv. Akad. Nauk SSSR Ser. Mat.* **22** (1958), 81–116. [Russian]
3. Z.-R. Guo, Approximation of continuous functions by Cesàro means of its Fourier series, *Sci. Sinica (Beijing)* **11** (1962), 1625–1634.
4. F. Móricz and B. E. Rhoades, Approximation by Nörlund means of double Fourier series for Lipschitz functions, *J. Approx. Theory*, in press.
5. F. Móricz and B. E. Rhoades, Approximation by Nörlund means of double Fourier series to continuous functions in two variables, *Constr. Approx.* in press.
6. S. B. Stechkin, On the order of the best approximations of continuous functions, *Izv. Akad. Nauk SSSR Ser. Mat.* **15** (1951), 219–242. [Russian]
7. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," Pergamon, Oxford, 1963.
8. I. E. Zhak, Concerning a theorem of L. Cesari on conjugate functions of two variables, *Dokl. Akad. Nauk SSSR* **87** (1952), 877–880. [Russian]
9. A. Zygmund, Smooth functions, *Duke Math. J.* **12** (1945), 47–76.
10. A. Zygmund, On the boundary values of functions of several complex variables, *Fund. Math.* **36** (1949), 207–235.
11. A. Zygmund, "Trigonometric Series," Vol. 1, Cambridge Univ. Press, Cambridge, 1959.